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Black Holes In $N = 2$ Supergravity Theories And Harmonic Functions

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ABSTRACT

We present dyonic BPS static black hole solutions for general $d = 4$, $N = 2$ supergravity theories coupled to vector and hypermultiplets. These solutions are generalisations of the spherically symmetric Majumdar-Papapetrou black hole solutions of Einstein-Maxwell gravity and are completely characterised by a set of constrained harmonic functions. In terms of the underlying special geometry, these harmonic functions are identified with the imaginary part of the holomorphic sections defining the special Kähler manifold and the metric is expressed in terms of the symplectic invariant Kähler potential. The relations of the holomorphic sections to the harmonic functions constitute the "*generalised stabilisation equations*" for the moduli fields. In addition to asymptotic flatness, the harmonic functions are also constrained by the requirement that the Kähler connection of the underlying Hodge-Kähler manifold has to vanish in order to obtain static solutions. The behaviour of these solutions near the horizon is also explained.

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1 Introduction

There has been lots of interest as well as progress in the study of BPS black holes in ungauged four-dimensional $N = 2$ supergravity theories coupled to vector and hypermultiplets [1]-[14]. The underlying special geometry structure of the $N = 2$ theory has been a very useful tool in the analysis of these black holes[2]. In a general $N = 2$ supersymmetric theory, the mass of a BPS state which breaks half of the supersymmetry is given in terms of the modulus of the central charge of the underlying $N = 2$ supersymmetry algebra [19]. The central charge is a function of the scalars of the theory as well as the electric and magnetic charges which correspond to the $U(1)^{n+1}$ gauge symmetry in a theory with n abelian vector multiplets.[3] Moreover, the black hole ADM mass in the $N = 2$ theory is governed by the value of the central charge at infinity and thus depends on the electric and magnetic charges as well as the asymptotic values of the scalar fields.

The recent work of Ferrara, Kallosh and Strominger [1, 2, 3] provided an algorithm for the macroscopic determination of Bekenstein-Hawking entropy [17] for the $N = 2$ extremal black holes. One simply extremizes the central charge at fixed values of electric and magnetic charges and the extremum value Z_h of the central charge gives the entropy S, which is quarter of the area A of the horizon, via the relation [3]

$$S = \frac{A}{4} = \pi |Z_h|^2 \tag{1}$$

Moreover, the horizon acts as an attractor for the scalar fields. This means that the values of the moduli at the horizon are fixed and completely independent of their values at spatial infinity. In general the fixed values for the scalars at the horizon are those which extremize the central charge. The near horizon physics also depends on topological data in the cases where the $N = 2$ supergravity models are obtained from compactifying type-II string theories on Calabi-Yau three-folds. The prepotential of these theories depend on the classical intersection numbers, Euler number and rational instanton numbers. Therefore, for these models, the central charge and the entropy depend on the electric and magnetic charges and the topological data of the Calabi-Yau manifold [7].

The above results were obtained by making use of the supersymmetry transformation rules for the gravitino and the gauginos in the ungauged bosonic part of $N = 2$ supergravity coupled to vector multiplets, where it was also assumed that the hypermultiplets take constant values. The condition for unbroken supersymmetry near the horizon is the statement that the holomorphic covariant derivative of the central charge (related to the graviphoton field strength) must vanish. At this point, one obtains equations which relate

[2]For a review on $N = 2$ supergravity and special geometry see for example [15]

[3]The extra $U(1)$ is due to the graviphoton.

the moduli to the magnetic and electric charges and other discrete parameters. Using the extremization procedure of the central charge [3], the entropy formulae, derived from the explicit solutions, for $N = 4$ and $N = 8$ extreme black hole [26] were obtained. The entropy of $N = 2$ black holes has been studied for particular models both at the classical, perturbative and non-perturbative level [4]-[12]. It should be mentioned that while the entropies for $N = 4, 8$ black holes are unique and are determined in terms of quantised magnetic and electric charges, those for the $N = 2$ theories depend on the specific details of the underlying special Kähler manifold.

A special class of black hole solutions, the so called double extreme black holes, is obtained if one assumes that the moduli fields take the same fixed value from the horizon to spatial infinity [4]. For these solutions, the central charge is constant everywhere and thus the black hole ADM mass coincides with the Bertotti-Robinson mass just as in the case of pure Einstein-Maxwell theory. The metric of the double extreme black hole is of the extreme Reissner-Nordstrøm form and can be expressed as

$$\begin{aligned} ds^2 &= \Big(1 + \frac{\sqrt{A/4\pi}}{r}\Big)^{-2} dt^2 - \Big(1 + \frac{\sqrt{A/4\pi}}{r}\Big)^{2} (d\vec{x})^2 \\ &= \Big(1 + \frac{|Z_h|}{r}\Big)^{-2} dt^2 - \Big(1 + \frac{|Z_h|}{r}\Big)^{2} (d\vec{x})^2, \end{aligned} \tag{2}$$

where $r \equiv \sqrt{\vec{x} \cdot \vec{x}}$, and $\vec{x}$ is the position vector in flat three-dimensional space referred to as the background space. The mass of the black hole is defined by

$$g_{tt} = (1 - \frac{2M_{ADM}}{r} + \cdots) \tag{3}$$

and from (2) one obtains

$$M_{ADM} = \sqrt{\frac{A}{4\pi}} = |Z_h|. \tag{4}$$

Therefore, the double extreme black hole solution is, in principle, known for all $N = 2$ supergravity theories and is completely specified in terms of the central charge of the underlying $N = 2$ supersymmetry algebra. Thus if we denote the covariantly holomorphic sections by (L^I, M_I)[4], then the central charge Z can be expressed by

$$Z = M_I p^I - L^I q_I, \tag{5}$$

where q_I and p^I are the electric and magnetic charges. The double extreme black hole is then completely determined by

$$Z_h = M_{hI} p^I - L^I_h q_I, \tag{6}$$

[4]These quantities are defined in the next section

where (L_h^I, M_{hL}) are the values of the covariantly holomorphic sections evaluated at the near horizon, and are given in terms of the "*stabilisation equations*"

$$\begin{aligned} i(Z_h\bar{L}^I - \bar{Z}_h L^I) &= p^I, \\ i(Z_h\bar{M}_I - \bar{Z}_h M_I) &= q_I, \end{aligned} \tag{7}$$

which are obtained from the extremization of the central charge.

The simplicity of the double extreme black holes comes from restricting the moduli fields to constant values everywhere. However, near the horizon, both of the extreme and the double extreme dyonic BPS black holes lose all their scalar hair and depend only on conserved charges and discrete parameters corresponding to gauge symmetries and topological data. The near horizon can be approximated by the Bertotti-Robinson type metric where the area of the black is interpreted as the mass of the Bertotti-Robinson universe [33]

$$\begin{aligned} ds^2 &= \frac{r^2}{M_{BR}^2}dt^2 - \frac{M_{BR}^2}{r^2}(d\vec{x})^2 \\ &= \frac{4\pi r^2}{A}dt^2 - \frac{A}{4\pi r^2}(d\vec{x})^2 \\ &= \frac{r^2}{|Z_h|^2}dt^2 - \frac{|Z_h|^2}{r^2}(d\vec{x})^2. \end{aligned} \tag{8}$$

The Bertotti-Robinson metric plays a special role in Einstein-Maxwell gravity in the sense that it can be considered as an alternative, maximally supersymmetric vacuum state. The extreme Reissner-Nordstrøm metric is a soliton which breaks half of the supersymmetry and interpolates between two maximally supersymmetric configurations; the trivial flat metric vacuum and the Bertotti-Robinson vacuum [16]. Maximally supersymmetric configurations are of course those where the full $N = 2$ supersymmetry is restored. For BPS black holes in $N = 2$ supergravity theories with vector multiplets, the near horizon metric, as for the pure supergravity case, is still of the Bertotti- Robinson type. The additional feature is that the unbroken supersymmetry near the horizon restricts the moduli to fixed discrete values independent of their initial values at spatial infinity.

General extreme purely magnetic $N = 2$ black hole solutions were derived in [1]. Also extreme solutions were constructed for the so called axion-free STU model associated with the special Kähler manifold $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)}$in [11] and for supergravity models based on the minimal coupling manifolds $\frac{SU(n,1)}{SU(n)\times U(1)}$ in [12]. In the construction of [12] it was realised that the general extreme black hole solutions can be constructed from a set

of constrained complex harmonic functions. Later in [13], general extreme static black holes solutions were derived for an arbitrary $N = 2$ supergravity theory coupled to vector and hypermultiplets.

In this work, our purpose is to study general dyonic BPS extreme black hole solutions, for non-constant complex values of the moduli, and for an arbitrary $N = 2$ supergravity model coupled to vector and hypermultiplets. In deriving these black hole solutions, we obtain "*generalised stabilisation equations*" expressing the values of the moduli in terms of the charges at any point in space-time. Using these explicit solutions, the near horizon physics namely, the results of [3] are rederived. As a simple illustration of our results, we derive the known dyonic Reissner-Nordstrøm black hole solutions of Einstein-Maxwell gravity using the special geometry formulation of pure $N = 2$ supergravity theory. Part of the results of this work were briefly presented in [13].

$N = 2$ supergravity theories with vector and hypermultiplets are fully defined in terms of the special geometry of the manifold spanned by the scalars of the vector multiplets. In the analysis of the black hole solutions, we will use the symplectic formulation of special geometry which does not depend on the existence of a holomorphic prepotential. This work is organised as follows. In the next section, some basics of special Kähler geometry and $N = 2$ supergravity coupled to vector multiplets are reviewed where we display formulae relevant for our later discussions. The extreme black hole solutions with vector multiplets are then considered in section three. Section four contains a re-derivation of the Reissner-Nordstrøm solution of pure Einstein-Maxwell theory using the framework of special geometry. In section five, we use our general solutions and verify that the near horizon is approximated by a Bertotti-Robinson universe, where the scalars are always given in terms of the charges, and with values which extremise the central charge. The central charge evaluated at the horizon gives the Bekestein-Hawking entropy. The last section contains a summary and a discussion on how our results can be extended to the construction of stationary solutions.

2 Special Geometry and $N = 2$ Supergravity

In recent years, special geometry has emerged as an essential structure in the study of $N = 2$ supergravity theories, the vacuum structure of superstrings, topological field theories and mirror symmetry in Calabi-Yau three-folds. More recently, special geometry provided a useful tool in the study of the quantum moduli space and obtaining exact solutions of low energy effective actions for rigid and local $N = 2$ supersymmetric Yang-Mills theories [15].

The concept of special Kähler geometry was first introduced to the physics literature

in the analysis of $N = 2$ supergravity models coupled to vector multiplets [25]. There special Kähler manifolds were defined by the coupling of n vector multiplets to $N = 2$ supergravity. The complex scalars z^i of the $N = 2$ vector multiplets coupled to supergravity are coordinates of a special Kähler manifold. An intrinsic definition of special Kähler geometry in terms of symplectic bundles was later given [22] in connection with the geometry of the moduli of Calabi-Yau spaces where special Kähler manifolds were associated with the moduli space of the Kähler or complex structure. Also a coordinate-independent description was given in [23, 24], where special geometry was derived from the constraints of the extended $N = 2$ supersymmetry in the non-linear sigma models associated with an arbitrary number n of vector multiplets of a four dimensional supergravity. Special Kähler manifolds are Kähler-Hodge manifolds, with an additional constraint on the curvature [25]

$$R_{ij^\star kl^\star} = g_{ij^\star} g_{kl^\star} + g_{il^\star} g_{kj^\star} - C_{ikp} C_{j^\star l^\star p^\star} g^{pp^\star}, \tag{9}$$

where $g_{ij^\star} = \partial_i \partial_{j^\star} K$, is the Kähler metric with K the Kähler potential and C_{ijk} is a completely symmetric covariantly holomorphic tensor. Kähler-Hodge manifolds are characterised by a $U(1)$ bundle whose first Chern class is equal to the Kähler class. This implies that, locally, the $U(1)$ connection can be represented by

$$Q = -\frac{i}{2}(\partial_i K dz^i - \partial_{i^\star} K d\bar{z}^{i^\star}). \tag{10}$$

An intrinsic definition of special Kähler manifold can be given [21]-[24] in terms of a flat $2n + 2$ dimensional symplectic bundle over the Kähler-Hodge manifold, with the covariantly holomorphic sections

$$\begin{aligned} V &= \begin{pmatrix} L^I \\ M_I \end{pmatrix}, \qquad I = 0, \cdots, n \\ D_{i^\star} V &= (\partial_{i^\star} - \frac{1}{2}\partial_{i^\star} K)V = 0, \end{aligned} \tag{11}$$

obeying the constraints [18]

$$i < V|\bar{V} >= i(\bar{L}^I M_I - L^I \bar{M}_I) = 1, \qquad < V, U_i >= 0 \tag{12}$$

where the symplectic inner product is understood to be taken with respect to the metric $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$U_i = D_i V = (\partial_i + \frac{1}{2}\partial_i K)V = \begin{pmatrix} f_i^I \\ h_{iI} \end{pmatrix}. \tag{13}$$

In general, D_i is the covariant derivative with respect to the Levi-Civita connection and the connection $\partial_i K$. Thus, for a generic field ϕ^i which transforms under the Kähler transformation, $K \to K + f + \bar{f}$, by the $U(1)$ transformation $\phi^i \to e^{-(\frac{p}{2}f + \frac{\bar{p}}{2}\bar{f})}\phi^i$, we have

$$D_i \phi^j = \partial_i \phi^j + \Gamma^j_{ik}\phi^k + \frac{p}{2}\partial_i K \phi^j. \tag{14}$$

One also defines the covariant derivative $D_{i^\star}$ in the same way but with p replaced with $\bar{p}$. The sections (L^I, M_I) has the weights $p = -\bar{p} = 1$ and C_{ijk} has the weights $p = -\bar{p} = 2$.

In general, one can write

$$\begin{aligned} M_I &= \mathcal{N}_{IJ} L^J, \\ h_{iI} &= \bar{\mathcal{N}}_{IJ} f_i^J. \end{aligned} \tag{15}$$

The complex symmetric $(n+1) \times (n+1)$ matrix $\mathcal{N}$ encodes the couplings of the vector fields in the corresponding $N = 2$ supergravity theory.

It can be shown [20]-[25] that the condition (9) can be obtained from the integrability conditions on the following differential constraints

$$\begin{aligned} D_i V &= U_i, \\ D_i U_j &= i C_{ijk} g^{kl^\star} \bar{U}_{l^\star}, \\ D_i \bar{U}_{j^\star} &= g_{ij^\star} \bar{V}, \\ D_i \bar{V} &= 0. \end{aligned} \tag{16}$$

It is well known that the constraints (16) can in general be solved in terms of a holomorphic function of degree two [25]. However, there exists symplectic sections for which such a holomorphic function does not exist. This, for example, appears in the study of the effective theory of the $N = 2$ heterotic strings [19]. Thus it is more natural to use the differential constraints (16) as the fundamental equations of special geometry.

The Kähler potential can be constructed in a symplectic invariant manner as follows. Define the sections Ω by

$$V = \begin{pmatrix} L^I \\ M_I \end{pmatrix} = e^{\frac{K}{2}}\Omega = e^{\frac{K}{2}} \begin{pmatrix} X^I \\ F_I \end{pmatrix}. \tag{17}$$

It immediately follows from (11) that Ω is holomorphic;

$$\partial_{i^\star} X^I = \partial_{i^\star} F_I = 0. \tag{18}$$

Using (12), one obtains

$$\begin{aligned} K &= -\log\Big(i < \Omega|\bar{\Omega} > \Big) \\ &= -\log\Big[i(\bar{X}^I F_I - X^I \bar{F}_I)\Big]. \end{aligned} \tag{19}$$

Exploiting the relations (12),(16) and (15), the following symplectic expressions can be derived for the Kähler metric and C_{ijk}

$$g_{ij^\star} = -i < U_i|\bar{U}_{j^\star} >= -2 f_i^I \mathrm{Im}\mathcal{N}_{IJ} \bar{f}_{j^\star}^J \tag{20}$$

$$C_{ijk} = < D_i U_j, U_k > . \tag{21}$$

For our purposes, it is also useful to display the following relations

$$g^{ij^\star} f_i^I \bar{f}_{j^\star}^J = -\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1IJ} - \bar{L}^I L^J, \qquad \mathrm{Im}\mathcal{N}_{IJ} L^I \bar{L}^J = -\frac{1}{2}. \tag{22}$$

It should be mentioned that the dependence of the gauge couplings on the scalars characterising homogenous special Kähler manifolds of $N = 2$ supergravity theory can also be determined from the knowledge of the corresponding embedding of the isometry group of the scalar manifold into the symplectic group à la Gaillard and Zumino [35, 36].

The $N = 2$ supergravity action includes one gravitational, n vector and hypermultiplets. However, for our purposes, we will neglect the hypermultiplets in what follows and assume that these fields are constants. In this case, the bosonic $N = 2$ action is given by

$$S_{N=2} = \int \sqrt{-g} d^4x \Big(-\frac{1}{2}R + g_{ij^\star} \partial^\mu z^i \partial_\mu \bar{z}^{j^\star} + i \left(\bar{\mathcal{N}}_{IJ} F^{-I}_{\mu\nu} F^{-J\mu\nu} - \mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu} \right) \tag{23}$$

where

$$F^{I\pm}_{\mu\nu} = \frac{1}{2}\Big(F^I_{\mu\nu} \pm \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma I} \Big) \tag{24}$$

The important field strength combinations which enter the chiral gravitino and gauginos supersymmetry transformation rules are given by

$$T^-_{\mu\nu} = M_I F^I_{\mu\nu} - L^I G_{I\mu\nu} = 2i(\mathrm{Im}\mathcal{N}_{IJ}) L^I F^{J-}_{\mu\nu} \tag{25}$$

$$G^{-i}_{\mu\nu} = -g^{ij^\star} \bar{f}^I_{j^\star} (\mathrm{Im}\mathcal{N})_{IJ} F^{J-}_{\mu\nu}. \tag{26}$$

The supersymmetry transformation for the chiral gravitino $\psi_{\alpha\mu}$ and gauginos $\lambda^{i\alpha}$ in a bosonic background of $N = 2$ supergravity are given by

$$\delta\, \psi_{\alpha\mu} = \nabla_\mu \epsilon_\alpha - \frac{1}{4} T^-_{\rho\sigma} \gamma^{\rho\sigma}\, \gamma_\mu\, \varepsilon_{\alpha\beta} \epsilon^\beta\, , \tag{27}$$

$$\delta\lambda^{i\alpha} = i\partial_\mu z^i \gamma^\mu \epsilon^\alpha + G^{-i}_{\rho\sigma} \gamma^{\rho\sigma} \epsilon^\beta \varepsilon^{\alpha\beta} \tag{28}$$

where ϵ_β is the chiral supersymmetry parameter, $\varepsilon^{\alpha\beta}$ is the $SO(2)$ Ricci tensor and the space-time covariant derivative ∇_μ also contains the Kähler connection

$$Q_\mu = -\frac{i}{2}\Big(\partial_i K \partial_\mu z^i - \partial_{i^\star} K \partial_\mu \bar{z}^{i^\star}\Big), \tag{29}$$

Therefore we have

$$\nabla_\mu \epsilon_\alpha = (\partial_\mu - \frac{1}{4} w_\mu^{ab} \gamma_{ab} + \frac{i}{2} Q_\mu)\epsilon_\alpha \tag{30}$$

where w_μ^{ab} is the spin connection.

The mass of a BPS state breaking half of the supersymmetry is given in terms of the central charge Z of the theory is defined by [19]

$$M_{BPS} = |Z|^2 = |M_I p^I - L^I q_I|^2 = e^K |F_I p^I - X^I q_I|^2 \tag{31}$$

where the electric and magnetic charges are defined by

$$\begin{aligned} q_I &= \int_{S^2_\infty} G_{I\mu\nu} dx^\mu \wedge dx^\nu \\ p^I &= \int_{S^2_\infty} F^I_{\mu\nu} dx^\mu \wedge dx^\nu \end{aligned} \tag{32}$$

where S^2_∞ is the two-sphere at infinity.

3 General Black Hole Solutions

In this section we derive general static BPS black hole solutions for $N = 2$ supergravity theory with an arbitrary number of vector multiplets. It is known [27, 28] that static solutions admitting supersymmetries are given by the Majumdar-Papapetrou black holes solutions [32]. Here we consider spherically symmetric solutions which can be written in the form

$$ds^2 = e^{2U(r)} dt^2 - e^{-2U(r)} (d\vec{x})^2, \tag{33}$$

In order to find the explicit BPS black hole solution it is more convenient to use the supersymmetry transformations rules of the fermi fields since these transformation rules depend linearly on the first derivatives of the bosonic fields and thus their vanishing provide first order differential equations [1, 34].

From the conditions of vanishing supersymmetry transformation, *i.e.*, $\delta\psi_{\alpha\mu} = \delta\lambda^{i\alpha} = 0$, one obtains, for a particular choice of the supersymmetry parameter, the following

first order differential equations [34, 13]

$$\frac{dU}{dr} = -2\mathrm{i}\,(\mathrm{Im}\mathcal{N})_{IJ}\,L^I t^J \frac{e^U}{r^2} \tag{34}$$

$$\frac{dz^i}{dr} = -2\mathrm{i}g^{ij^\star}\bar{f}^I_{j^\star}\,(\mathrm{Im}\mathcal{N})_{IJ}\,\frac{t^J}{r^2}e^U \tag{35}$$

where

$$t^I = \frac{1}{2}\Big(p^I - i(\mathrm{Im}\mathcal{N}^{-1})^{IK}(\mathrm{Re}\mathcal{N})_{KM}p^M + i(\mathrm{Im}\mathcal{N}^{-1})^{IK}q_K\Big). \tag{36}$$

The solution of the above equations of course depends on the particular model one is considering, *i.e.*, the choice of the special Kähler manifold. In what follows we will solve the above differential equations in a model independent way. To begin, consider the first differential equation (34). This can be rewritten as

$$\frac{dU}{dr} = -Z\frac{e^U}{r^2} = e^{\frac{K}{2}}(X^I q_I - F_I p^I)\frac{e^U}{r^2}. \tag{37}$$

Our ansatz for the solution is to take

$$e^{-2U} = e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I), \tag{38}$$

which is the most natural choice due to the symplectic invariance of the Kähler potential. This choice enables us to rewrite (37) in the following form

$$\frac{d}{dr}e^{-2U} = \frac{d}{dr}e^{-K} = -2\frac{(X^I q_I - F_I p^I)}{r^2}. \tag{39}$$

Differentiating e^{-K} with respect to r, we get

$$\frac{d}{dr}e^{-K} = i(\frac{d\bar{X}^I}{dr}F_I + \bar{X}^I\frac{dF_I}{dr} - \frac{dX^I}{dr}\bar{F}_I - X^I\frac{d\bar{F}_I}{dr}) \tag{40}$$

and demanding that our solutions satisfy,

$$\bar{X}^I\frac{dF_I}{dr} - \frac{dX^I}{dr}\bar{F}_I = \frac{d\bar{X}^I}{dr}F_I - X^I\frac{d\bar{F}_I}{dr}, \tag{41}$$

then from (39) we obtain

$$\frac{d}{dr}e^{-2U} = 2i\Big(\frac{d\bar{X}^I}{dr}F_I - X^I\frac{d\bar{F}_I}{dr}\Big). \tag{42}$$

If we write

$$i(X^I - \bar{X}^I) = f^I(r), \qquad i(F_I - \bar{F}_I) = g_I(r) \tag{43}$$

Then (42) can be rewritten in the form

$$\frac{d}{dr}e^{-2U} = 2\Big(X^I\frac{dg_I}{dr} - F_I\frac{df^I}{dr}\Big) \tag{44}$$

where we have made use of the relation imposed by the underlying special geometry,

$$X^I\partial_r F^I - F^I\partial_r X^I = 0, \tag{45}$$

which directly follows from the second relation in (12).

Now comparing (44) with (39) we immediately arrive at the following result

$$f^I(r) = \tilde{h}^I + \frac{p^I}{r}, \qquad g_I(r) = h_I + \frac{q_I}{r}. \tag{46}$$

where h_I and $\tilde{h}^I$ are constants which must obey the constraints coming from demanding the asymptotic flatness of our solution as well the condition (41). We now turn to show that the solution specified by (38) and (46) also solves the differential equation (35) obtained from the vanishing of the gauginos supersymmetry transformation.

If we multiply both sides of (35) by f_i^J and use (22), then the following equation is obtained.

$$f_i^I\frac{dz^i}{dr} = it^I\frac{e^U}{r^2} + \bar{L}^I Z\frac{e^U}{r^2}, \tag{47}$$

from which one can derive the following two relations

$$\frac{e^U}{r^2}(Z\bar{L}^I - \bar{Z}L^I) \;=\; -ip^I\frac{e^U}{r^2} + 2i\mathrm{Im}(f_i^I\frac{dz^i}{dr}) \tag{48}$$

$$\frac{e^U}{r^2}(Z\bar{M}_I - \bar{Z}M_I) \;=\; -iq_I\frac{e^U}{r^2} + 2i\mathrm{Im}(h_{iI}\frac{dz^i}{dr}) \tag{49}$$

where we remind the reader that

$$\begin{aligned} f_i^I &= (\partial_i + \frac{1}{2}\partial_i K)L^I \\ h_{iI} &= (\partial_i + \frac{1}{2}\partial_i K)M_I. \end{aligned}$$

To evaluate the right hand side of (48) in terms of the symplectic sections, we first note the following relation

$$\partial_i K\frac{dz^i}{dr} = ie^K(\bar{F}_I\partial_r X^I - \bar{X}^I\partial_r F_I) \tag{50}$$

which upon using (43) and (45), can be rewritten as

$$\partial_i K \frac{dz^i}{dr} = e^K (\bar{F}_I \partial_r f^I - \bar{X}^I \partial_r g_I). \tag{51}$$

Using (51) we get

$$\begin{aligned} f_i^I \frac{dz^i}{dr} &= (\partial_i + \frac{1}{2}\partial_i K) L^I \frac{dz^i}{dr} \\ &= \partial_i K \frac{dz^i}{dr} L^I + e^{\frac{K}{2}} \frac{dX^I}{dr} \\ &= e^K (\bar{F}_J \partial_r f^J - \bar{X}^J \partial_r g_J) L^I + e^{\frac{K}{2}} \frac{dX^I}{dr} \end{aligned} \tag{52}$$

Substituting the relation (52) into equation (48) gives the following differential equation

$$\begin{aligned} &\frac{e^{U+\frac{K}{2}}}{r^2} \Big[(F_J p^J - X^J q_J) \bar{L}^I - (\bar{F}_J p^J - \bar{X}^J q_J) L^I \Big] \\ &= -ip^I \frac{e^U}{r^2} + e^K \Big[L^I (\bar{F}_J \frac{df^J}{dr} - \bar{X}^J \frac{dg_J}{dr}) - \bar{L}^I (F_J \frac{df^J}{dr} - X^J \frac{dg_J}{dr}) \Big] - ie^{\frac{K}{2}} \frac{df^I}{dr} \end{aligned} \tag{53}$$

Similar manipulation for (49) leads to

$$\begin{aligned} &\frac{e^{U+\frac{K}{2}}}{r^2} \Big[(F_J p^J - X^J q_J) \bar{M}_I - (\bar{F}_J p^J - \bar{X}^J q_J) M_I \Big] \\ &= -iq_I \frac{e^U}{r^2} + e^K \Big[M_I (\bar{F}_J \frac{df^J}{dr} - \bar{X}^J \frac{dg_J}{dr}) - \bar{M}_I (F_J \frac{df^J}{dr} - X^J \frac{dg_J}{dr}) \Big] - ie^{\frac{K}{2}} \frac{dg_I}{dr} \end{aligned} \tag{54}$$

The above rather ugly equations can be easily seen to be solved by (38) and (46).

In deriving our solutions the condition (41) was imposed on the holomorphic sections. This restricts the allowed values of the harmonic functions defining the black hole solutions. In addition, the harmonic functions are normalised in order to obtain asymptotically flat solutions. The constraint (41) is the vanishing of the Kähler connection Q_μ of the underlying Kähler-Hodge manifold which is essential if we were to obtain static black hole solutions. To explain, we rewrite the Kähler connection,

$$Q_\mu = -\frac{i}{2} \Big(\partial_i K \partial_\mu z^i - \partial_{i^\star} K \partial_\mu \bar{z}^{i^\star} \Big)$$

in the following form

$$Q_\mu = -\frac{1}{2} e^K \Big(\bar{X}^I \partial_\mu F_I - \partial_\mu X^I \bar{F}_I - \partial_\mu \bar{X}^I F_I + X^I \partial_\mu \bar{F}_I). \tag{55}$$

Clearly (41) follows from $Q_\mu = 0$. In order to find the explicit additional constraints on the constants of the harmonic functions, it is more convenient to express Q_μ in terms of $f^I(r)$ and $g_I(r)$. In terms of these quantities, the Kähler connection takes a very simple form,

$$Q_\mu = -\frac{e^K}{2}(f^I \partial_\mu g_I - g_I \partial_\mu f^I) \tag{56}$$

For our solutions, where $f^I(r)$ and $g_I(r)$ are given by harmonic functions, the vanishing of the Kähler connection does not impose any restrictions on the electric and magnetic charges. However, the values of h_I and $\tilde{h}^I$, related to the values of the scalars at infinity are constrained by the following conditions

$$\tilde{h}^I q_I - h_I p^I = 0. \tag{57}$$

This condition implies that the central charge of the supersymmetry algebra must be real for our solution. Therefore, the vanishing of the Kähler connections is related to the reality of the central charge. This can be easily seen by noting that for our solutions

$$\begin{aligned} Z - \bar{Z} &= e^{\frac{K}{2}}\Big((F_I - \bar{F}_I)p^I - (X^I - \bar{X}^I)q_I\Big) \\ &= ie^{\frac{K}{2}}\Big(\tilde{h}^I q_I - h_I p^I\Big) = 0. \end{aligned} \tag{58}$$

Clearly for the static solutions, the central charge has to be real in order for the differential equation (37) to make sense.

To summarise we have found static BPS black hole solutions for $N = 2$ supergravity coupled to vector multiplets. These solutions are given by

$$\begin{aligned} ds^2 &= e^K dt^2 - e^{-K} d\vec{x}^2, \\ &= -\frac{i}{(\bar{X}^I F_I - X^I \bar{F}_I)} dt^2 - i(\bar{X}^I F_I - X^I \bar{F}_I) d\vec{x}^2, \end{aligned} \tag{59}$$

where

$$i(X^I - \bar{X}^I) = \tilde{h}^I + \frac{p^I}{r}, \qquad i(F_I - \bar{F}_I) = h_I + \frac{q_I}{r} \tag{60}$$

and

$$\tilde{h}^I q_I - h_I p^I = 0, \qquad e^K_\infty = 1. \tag{61}$$

Eq. (60) provides us with the "*generalised stabilisation equations*" which express the values of the moduli at any point in space-time.

4 Reissner-Nordstrøm Solutions From Special Geometry

As an application, we use our general solutions and consider the simplest example, i.e., Einstein-Maxwell gravity and derive the extreme Reissner-Nordstrøm black hole solution of this theory using the framework of special geometry. In this case, there are no vector multiplets and the only scalar present is that of the gravitational multiplet which contains the graviphoton.

To start, consider the spherically symmetric Majumdar-Papapetrou metric

$$ds^2 = V^{-2}(r)dt^2 - V^2(r)(d\vec{x})^2. \tag{62}$$

It is well known that the source-free Einstein-Maxwell equations of motion for the metric in (62) reduce to Laplace's equation for $V(r)$

$$\nabla^2 V(r) = 0. \tag{63}$$

The Majumdar-Papapetrou black hole solutions for Einstein-Maxwell theory are known to admit supersymmetry [27, 28]. Therefore, we can imbed Einstein-Maxwell theory in $N = 2$ supergravity and view the metric (62) as a solution which breaks half of the supersymmetry. From our previous discussion, the scale V^2 can be identified with e^{-K}. If we describe the pure $N = 2$ supergravity theory by the holomorphic prepotential $F(X^0) = -\frac{i}{2}(X^0)^2$, we obtain

$$V^2 = e^{-K} = i(\bar{X}^0 F_0 - X^0 \bar{F}_0) = 2X^0\bar{X}^0 \tag{64}$$

and thus our black hole solution can be expressed by

$$ds^2 = \frac{1}{2X^0\bar{X}^0}dt^2 - 2X^0\bar{X}^0(d\vec{x})^2, \tag{65}$$

with

$$\begin{aligned} i(X^0 - \bar{X}^0) &= \tilde{h}^0 + \frac{p^0}{r}, \\ (X^0 + \bar{X}^0) &= h_0 + \frac{q_0}{r} \end{aligned} \tag{66}$$

where q_0, p^0 are the electric and magnetic quantum charges of the $U(1)$ gauge group associated with the graviphoton. Now for a static solution with both charges present, X^0 is complex and is given by[5]

$$X^0 = \frac{1}{2}(h_0 + \frac{q_0}{r}) - i\frac{1}{2}(\tilde{h}^0 + \frac{p^0}{r}). \tag{67}$$

[5] An important point is to notice that the gauge choice $X^0 = 1$ is not convenient for the study of black hole solutions of $N = 2$ supergravity.

The conditions of asymptotic flatness and the vanishing of the Kähler connection gives the following conditions

$$\tilde{h}^0 q_0 - h_0 p^0 = 0, \qquad (\tilde{h}^0)^2 + (h_0)^2 = 2. \tag{68}$$

This fixes completely the values of h_0 and $\tilde{h}^0$ to

$$\tilde{h}^0 = \sqrt{2}\frac{p^0}{\sqrt{p_0^2 + q_0^2}}, \qquad h_0 = \sqrt{2}\frac{q_0}{\sqrt{p_0^2 + q_0^2}} \tag{69}$$

and the metric (65) can be written as

$$ds^2 = \Big(1 + \frac{1}{r}\sqrt{\frac{p_0^2 + q_0^2}{2}}\Big)^{-2} dt^2 - \Big(1 + \frac{1}{r}\sqrt{\frac{p_0^2 + q_0^2}{2}}\Big)^2 (d\vec{x})^2, \tag{70}$$

which is the extreme Reissner-Nordstrøm black hole solution of Einstein-Maxwell gravity.

5 Entropy and Minimal Central Charge

In this section, the well known behaviour of static extreme black holes at the near horizon is rederived and confirmed using our explicit general solutions.

Near the horizon, the constants $\tilde{h}^I$ and h_I drop out and the metric scale e^{-K_h} can be approximated as follows

$$\begin{aligned}
e^{-K_h} &= i(\bar{X}_h^I F_{Ih} - X_h^I \bar{F}_{Ih}) \\
&= i\Big[(X_h^I + i\frac{p^I}{r})F_{Ih} - X_h^I(F_{Ih} + i\frac{q_I}{r})\Big] \\
&= \frac{1}{r}(X_h^I q_I - F_{Ih} p^I) \\
&= -\frac{1}{r} Z_h e^{\frac{-K_h}{2}}.
\end{aligned}$$

This implies that the near horizon metric takes the Bertotti-Robinson form

$$\begin{aligned}
ds^2 &= \frac{r^2}{M_{BR}^2} dt^2 - \frac{M_{BR}^2}{r^2}(d\vec{x})^2 \\
&= \frac{r^2}{Z_h^2} dt^2 - \frac{Z_h^2}{r^2}(d\vec{x})^2.
\end{aligned} \tag{71}$$

Next, near the horizon, the "*generalised stabilisation equations*" expressed in terms of the covariantly holomorphic sections, reduce to the following equations

$$ie^{-K_h/2}(L^I - \bar{L}^I)_h = \frac{p^I}{r} \quad (72)$$

$$ie^{-K_h/2}(M_I - \bar{M}_I)_h = \frac{q_I}{r} \quad (73)$$

Using the relation $e^{-K_h/2} = -\frac{Z_h}{r}$, one obtains

$$iZ_h(\bar{L}_h - L_h) = p^I \quad (74)$$

$$iZ_h(\bar{M}_{Ih} - M_{Ih}) = q^I \quad (75)$$

Which are the stabilisation conditions found in [3]. This also means that, for our solutions, the central charge is extremum at the horizon

$$(D_i Z)_h = 0. \quad (76)$$

Clearly, at the horizon, the central charge (entropy) and the values of the scalar fields are completely independent of the values of the scalar fields at spatial infinity which agrees with the results of [1, 2, 3].

6 Discussions

In this work we derived general $N = 2$ static black hole solutions for ungauged $N = 2$ supergravity theories coupled to an arbitrary number of vector and hypermultiplets. The solutions found are spherically symmetric Majumdar-Papapetrou like metrics [32] and are entirely expressed in terms of the Kähler potential of the underlying special Kähler manifold spanned by the scalars of the vector multiplets. For these solutions, the imaginary part of the holomorphic sections are given by a set of constrained harmonic functions which depend on the electric and magnetic charges. This is not surprising since the $N = 2$ supergravity theory can be fully constructed out of the holomorphic sections. Therefore, one should be able to express the black hole solutions in terms of symplectic invariants of the underlying special Kähler manifold. For our static solutions, the symplectic invariant is simply the Kähler potential. This implies that any perturbative or non-perturbative corrections to our black hole solutions can be understood in terms of corrections to the Kähler potential of the scalar fields in the theory.

It was also found that the ansatz for the static solutions forces one to set the Kähler connection to zero. Thus one expects the Kahler connection (which is also symplectic

invariant) to play a role in the construction of stationary solutions. We will report on the stationary solutions in a separate publication. The role of the Kähler connection can be easily seen from the following simple observation. It is known from the work of Tod [28] that the most general form of the metric admitting supersymmetries can be written in the form

$$ds^2 = V\bar{V}(dt + \vec{w}.\vec{dx})^2 - (V\bar{V})^{-1}(d\vec{x})^2 \tag{77}$$

where (in the absence of dust), V is the inverse of a harmonic function and w is defined by

$$\vec{\nabla} \times \vec{w} = -\frac{i}{(V\bar{V})^2}(\bar{V}\vec{\nabla}V - V\vec{\nabla}\bar{V}). \tag{78}$$

These solutions constitute a class of stationary metrics known as "*conformastationary*" and were discovered by Neugebauer [29], Perjés [30] and Israel and Wilson [31]. To see how the symplectic invariant Kähler connection is related to w, we note that if we write $e^K = V\bar{V}$, with V time independent, then the Kahler connection vector becomes

$$\begin{aligned} \vec{Q} &= -\frac{i}{2V\bar{V}}(\bar{V}\vec{\nabla}V - V\vec{\nabla}\bar{V}) \\ &= \frac{1}{2}(V\bar{V})(\vec{\nabla} \times \vec{w}). \end{aligned} \tag{79}$$

Finally, we mention that while the constraints on the constants $(h_I, \tilde{h}^I)$ fix them completely for the case of pure Einstein-Maxwell gravity, in the presence of vector multiplets one still have freedom in choosing these constants. This allows one to study massless black holes configurations and the interplay between space-time and moduli-space singularities [37, 38]. This subject is currently under investigation.

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